Model Answer / Suggestive Answer

B.Sc. (Hon's) (Fifth Semester) Exam. 2013
Mathematics (Hydrodynamics)

AS – 2831

1 (i). Lagrangian method:- In this method we study the history of each fluid particle i.e. any fluid particle is selected and is pursued on its onward course observing the changes in velocity, pressure and density at each point and at each instant.

$$x = f_1(x_0, y_0, z_0, t),\ y = f_2(x_0, y_0, z_0, t),\ z = f_3(x_0, y_0, z_0, t).$$

(ii) Stream line or line of flow:- Stream line is the curve drawn on the fluid surface s.t. tangent at any point is along the direction of motion. Let q is the velocity of fluid and $\frac{d\vec{r}}{dt}$ be tangent vector. Then

$$q \times \frac{d\vec{r}}{dt} = 0 \Rightarrow \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}.$$

(iii) Stream tube and stream filament:- If we draw the stream lines from each point of a closed curve in the fluid, we obtain a tube called the stream tube.

A stream tube of infinitesimal cross-section is known as a stream filament.

P.T.O.

(iv) Bernoulli's Theorem:- When the motion is steady and the velocity potential does not exist we have

$$\frac{1}{2}q^2 + V + \int \frac{dp}{\rho} = c$$

where V is the force potential from which the external forces are derivable.

(v). Cauchy-Riemann equation in Polar form:-

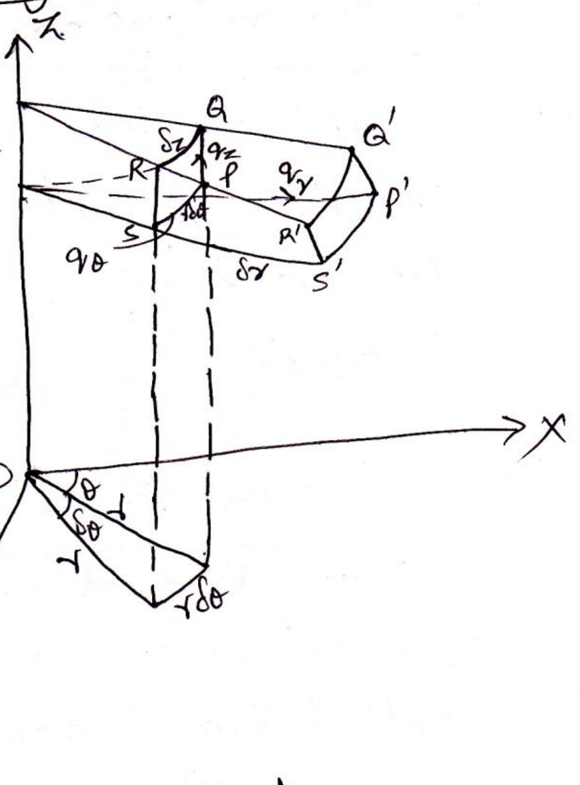
$$\frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \text{ and } \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r}.$$

(vi). (a)

(vii). (a)

(viii). True.

2. Equation of continuity in cylindrical co-ordinates:-



Let ρ be the density of the fluid at P at any time t.

$SS' = \delta r$, arc $SP = r\delta\theta$ and $PQ = \delta z$

Let q_r, q_θ, q_z be the velocity in the direction of the elements SS', arc SP and PQ.

Mass of the fluid passes through the faces PSRQ

$= \rho . r\delta\theta\, \delta z . q_r$ per unit time

$= f(r, \theta, z) \longrightarrow (1)$

and the mass of the fluid that passes out through face $P'S'R'Q'$ $= f(r+\delta r, \theta, z)$ per unit time

$= f(r,\theta,z) + \delta r \frac{\partial}{\partial r} f(r,\theta,z) + \text{------} \quad (2)$

$$\text{Excess of flow} = f(r,\theta,z) - \left[f(r,\theta,z) + \delta r \frac{\partial}{\partial r} f(r,\theta,z) + \cdots\right] \quad (3)$$

$$= -\delta r \frac{\partial}{\partial r} f(r,\theta,z)$$

$$= -\delta r \frac{\partial}{\partial r}(\rho r\,\delta\theta\,\delta z\, q_r)$$

$$= -\delta r\,\delta\theta\,\delta z \frac{\partial}{\partial r}(\rho r q_r) \longrightarrow (3)$$

Similarly, excess of flow in over flow out (SRR'S' and QPP'Q')

$$= -\delta r\,\delta\theta\,\delta z \cdot \frac{\partial}{\partial \theta}(\rho q_\theta) \longrightarrow (4)$$

and excess of flow in over flow-out along (PSS'P' and QRR'Q')

$$= -r\,\delta r\,\delta\theta\,\delta z \frac{\partial}{\partial z}(\rho q_z) \longrightarrow (5)$$

$\therefore$ Total rate of mass flow into the chosen element

$$= -\delta r\,\delta\theta\,\delta z\left[\frac{\partial}{\partial r}(r\rho q_r) + \frac{\partial}{\partial \theta}(\rho q_\theta) + r\frac{\partial}{\partial z}(\rho q_z)\right] \longrightarrow (6)$$

Also total mass of the fluid in the parallelopiped

$$= \rho r\,\delta r\,\delta\theta\,\delta z$$

$\Rightarrow$ Total rate of mass increase within the element

$$= \frac{\partial}{\partial t}(\rho r\,\delta r\,\delta\theta\,\delta z)$$

$$= r\,\delta r\,\delta\theta\,\delta z\,\frac{\partial \rho}{\partial t} \longrightarrow (7)$$

per unit time.

Then by the law of conservation of the fluid mass

$$r\,\delta r\,\delta\theta\,\delta z\,\frac{\partial \rho}{\partial t} = -\delta r\,\delta\theta\,\delta z\left[\frac{\partial}{\partial r}(\rho r q_r) + \frac{\partial}{\partial \theta}(\rho q_\theta) + r\frac{\partial}{\partial z}(\rho q_z)\right]$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r q_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho q_\theta) + \frac{\partial}{\partial z}(\rho q_z) = 0$$

Ans. 3 Euler's Equation of motion in vector form.-

Let a closed surface S enclosing the fluid be moving with the fluid, so that S contains the same fluid particles at any time.

Take any point P inside S. Let ρ be the density of fluid at P and δv be the elementary volume enclosing P, $\vec{q}$ be the velocity of fluid.

The momentum M of the volume V in S is given by
$$M = \int_V q \rho \, dv$$

rate of change of momentum is
$$\frac{dM}{dt} = \int_V \frac{dq}{dt} \rho \, dv + \int_V q \cdot \frac{d}{dt}(\rho \, dv)$$
$$= \int_V \frac{dq}{dt} \rho \, dv \quad \left(\because \int_V q \cdot \frac{d}{dt}(\rho \, dv) = 0 \right)$$

$\therefore$ Total force on the liquid in $V = \int_V F \rho \, dv$ (where F is external force per unit mass)

Then the total force on the surface
$$= -\int_S p \cdot n \, ds$$
$$= -\int_V \nabla p \, dv$$

$\therefore$ Rate of change of momentum = total force acting on the mass.
$$\int_V \frac{dq}{dt} \rho \, dv = \int_V F \rho \, dv - \int_V \nabla p \, dv$$
$$= \int_V (\rho F - \nabla p) \, dv$$
$$\Rightarrow \int_V \left(\rho \frac{dq}{dt} - \rho F + \nabla p \right) dv = 0$$
$$\Rightarrow \rho \frac{dq}{dt} - \rho F + \nabla p = 0$$
$$\Rightarrow \frac{dq}{dt} = F - \frac{1}{\rho} \nabla p \longrightarrow (1)$$

<u>Ques:- 4</u> — <u>Lagrange's Equation:-</u> Let $P(a,b,c)$ be initial position of particle and $Q(x,y,z)$ be position of same particle at time t where a, b, c, t are independent variable. We have to find x, y, z in terms of a, b, c and t. The velocity components along x, y, z are
$$u = \frac{dx}{dt}, \quad v = \frac{dy}{dt}, \quad w = \frac{dz}{dt} \quad \text{with acceleration } \frac{d^2x}{dt^2},$$
$\frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}$ along x, y, z axis.

P.T.O.

Let V be the the force potential for the external forces, then $X = -\frac{\partial V}{\partial x}$, $Y = -\frac{\partial V}{\partial y}$, $Z = -\frac{\partial V}{\partial z}$

where X, Y, Z be components of force along x, y, z axis. Then by Euler's dynamical equation of motion, we have

$$\frac{\partial u}{\partial t} = -\frac{\partial V}{\partial x} - \frac{1}{e}\frac{\partial p}{\partial x} \longrightarrow (1)$$

$$\frac{\partial v}{\partial t} = -\frac{\partial V}{\partial y} - \frac{1}{e}\frac{\partial p}{\partial y} \longrightarrow (2)$$

$$\frac{\partial w}{\partial t} = -\frac{\partial V}{\partial z} - 1/e\,\frac{\partial p}{\partial z} \longrightarrow (3)$$

Putting the value of u, v, w in (1), (2), (3), we get

$$\frac{\partial^2 x}{\partial t^2} = -\frac{\partial V}{\partial x} - \frac{1}{e}\frac{\partial p}{\partial x} \longrightarrow (4)$$

$$\frac{\partial^2 y}{\partial t^2} = -\frac{\partial V}{\partial y} - 1/e\,\frac{\partial p}{\partial y} \longrightarrow (5)$$

$$\frac{\partial^2 z}{\partial t^2} = -\frac{\partial V}{\partial z} - \frac{1}{e}\frac{\partial p}{\partial z} \longrightarrow (6)$$

Multiply eqn (4), (5), (6) by $\frac{\partial x}{\partial a}, \frac{\partial y}{\partial a}, \frac{\partial z}{\partial a}$ and then add.

$$\frac{\partial^2 x}{\partial t^2}\cdot\frac{\partial x}{\partial a} + \frac{\partial^2 y}{\partial t^2}\cdot\frac{\partial y}{\partial a} + \frac{\partial^2 z}{\partial t^2}\cdot\frac{\partial z}{\partial a} = -\frac{\partial V}{\partial a} - \frac{1}{e}\cdot\frac{\partial p}{\partial a}$$

Similarly, we obtain

$$\frac{\partial^2 x}{\partial t^2}\cdot\frac{\partial x}{\partial b} + \frac{\partial^2 y}{\partial t^2}\cdot\frac{\partial y}{\partial b} + \frac{\partial^2 z}{\partial t^2}\cdot\frac{\partial z}{\partial b} = -\frac{\partial V}{\partial b} - 1/e\cdot\frac{\partial p}{\partial b}$$

$$\frac{\partial^2 x}{\partial t^2}\cdot\frac{\partial x}{\partial c} + \frac{\partial^2 y}{\partial t^2}\cdot\frac{\partial y}{\partial c} + \frac{\partial^2 z}{\partial t^2}\cdot\frac{\partial z}{\partial c} = -\frac{\partial V}{\partial c} - 1/e\cdot\frac{\partial p}{\partial c}$$

These equations, together with the Lagrange's equation of continuity

$$\boxed{e\,\frac{\partial(x, y, z)}{\partial(a, b, c)} = 0}$$

P.T.O.

Ques. 5:- $\because$ motion is irrotational

$\therefore$ velocity potential exists and satisfy Laplace equation i.e. $\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0 \longrightarrow (1)$

Also $q^2 = \left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2 \longrightarrow (2)$

Diff. eqn (2) partially w.r.t. x and y, we get

$$q.\frac{\partial q}{\partial x} = \frac{\partial\phi}{\partial x}.\frac{\partial^2\phi}{\partial x^2}+\frac{\partial\phi}{\partial y}.\frac{\partial^2\phi}{\partial x\partial y} \longrightarrow (3)$$

and $$q.\frac{\partial q}{\partial y} = \frac{\partial\phi}{\partial y}.\frac{\partial^2\phi}{\partial y\partial x}+\frac{\partial\phi}{\partial y}.\frac{\partial^2\phi}{\partial y^2} \longrightarrow (4)$$

Again diff. (3) and (4) w.r.t. x and y.

$$q.\frac{\partial^2 q}{\partial x^2}+\left(\frac{\partial q}{\partial x}\right)^2 = \left(\frac{\partial^2\phi}{\partial x^2}\right)^2+\frac{\partial\phi}{\partial x}.\frac{\partial^3\phi}{\partial x^3}+\left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2+\frac{\partial\phi}{\partial y}.\frac{\partial^3\phi}{\partial x^2\partial y} \longrightarrow (5)$$

and $$q.\frac{\partial^2 q}{\partial y^2}+\left(\frac{\partial q}{\partial y}\right)^2 = \left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2+\frac{\partial\phi}{\partial x}.\frac{\partial^3\phi}{\partial y^2\partial x}+\left(\frac{\partial^2\phi}{\partial y^2}\right)^2+\frac{\partial\phi}{\partial y}.\frac{\partial^3\phi}{\partial y^3} \longrightarrow (6)$$

Adding (5) and (6)

$$q.\nabla^2 q = \left(\frac{\partial^2\phi}{\partial x^2}\right)^2+2\left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2+\left(\frac{\partial^2\phi}{\partial y^2}\right)^2 \quad (\text{from } (1))$$

$$= 2\,\frac{\partial^2\phi}{\partial x^2}+2\left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2 \longrightarrow (7)$$

Again, squaring and adding (3) and (4), we get

$$q^2\left[\left(\frac{\partial q}{\partial x}\right)^2+\left(\frac{\partial q}{\partial y}\right)^2\right] = q^2\left[\left(\frac{\partial^2\phi}{\partial x^2}\right)^2+\left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2\right] \quad (\text{from } (2) \text{ and } (1)) \longrightarrow (8)$$

from (7) and (8), we get

$$q.\nabla^2 q = \left(\frac{\partial q}{\partial x}\right)^2+\left(\frac{\partial q}{\partial y}\right)^2$$

Ques. 6 – Given $\vec{q} = K^2\frac{(xj - yi)}{x^2+y^2}$

Let $\vec{q} = ui + vj + wk$

then $u = \frac{-K^2 y}{x^2+y^2}$, $v = \frac{K^2 x}{x^2+y^2}$, $w=0$

$\because \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0 \quad \therefore$ eqn of continuity satisfy

So the motion is possible.

Now eqn of the stream lines are

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \Rightarrow x^2+y^2 = c_1 \longrightarrow (1) \text{ and } z = c_2 \longrightarrow (2)$$

P.T.O.

Again curl $q = 0$

Hence the flow is of the potential kind and we can find velocity potential ϕ s.t.

$\vec{q} = -\nabla\phi$. Thus $\frac{\partial\phi}{\partial x} = -u = \frac{k^2 y}{x^2+y^2} \to (3)$, $\frac{\partial\phi}{\partial y} = -v = \frac{-k^2 x}{x^2+y^2} \to (4)$, $\frac{\partial\phi}{\partial z} = -w = 0 \to (5)$

integrating (3), we get

$$\phi(x,y) = k^2 \tan^{-1}\frac{x}{y} + f(y) \longrightarrow (6)$$

$$\frac{\partial\phi}{\partial y} = -\frac{k^2 x}{x^2+y^2} + f'(y) \longrightarrow (7)$$

from (4) and (7), $f'(y) = 0 \Rightarrow f(y)$ is constt.

$\therefore \phi(x,y) = k^2 \tan^{-1}\left(\frac{x}{y}\right)$.

Ques.7 :- If ρ be the density of the gas at a distance x, and u is the velocity. Then according to given condition $\rho = \rho_0 \phi(vt-x) \longrightarrow (1)$

and eqn. of continuity is $\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0$

$$\Rightarrow \frac{\partial\rho}{\partial t} + \rho\frac{\partial u}{\partial x} + u\frac{\partial\rho}{\partial x} = 0 \longrightarrow (2)$$

Diff. (1) w.r.t. t and x.

$$\frac{\partial\rho}{\partial t} = \rho_0 v \phi'(vt-x) \text{ and } \frac{\partial\rho}{\partial x} = -\rho_0\phi'(vt-x)$$

Putting above value in (2), we get

$$\rho_0 v\phi'(vt-x) - \rho_0\phi'(vt-x) + \rho\frac{\partial u}{\partial x} = 0$$

$\Rightarrow \frac{du}{v-u} + \frac{\phi'(vt-x)}{\phi(vt-x)}dx = 0$ integrating

$$-\log(v-u) - \log\phi(vt-x) = \text{constt.}$$

or $(v-u)\cdot\phi(vt-x) = A$

But initially when $x=0$, $u=u_0$ then $A = (v-u_0)\phi(vt)$

$\therefore (v-u)\phi(vt-x) = (v-u_0)\phi(vt)$

$$\Rightarrow u = v + \frac{(u_0-v)\phi(vt)}{\phi(vt-x)}$$

P.T.O.

Ques. 8:– Given $u = 2cxy$, $v = c(a^2+x^2-y^2)$ $\longrightarrow$ (1)

Now $\frac{\partial u}{\partial x} = 2cy$, $\frac{\partial v}{\partial y} = -2cy$.

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ $\longrightarrow$ (2) i.e. equation of continuity satisfied

Hence u and v constitute a possible fluid motion.

Let ψ be the required stream function. Then

$$u = -\frac{\partial \psi}{\partial y} \quad \text{or} \quad \frac{\partial \psi}{\partial y} = -2cxy \longrightarrow (3)$$

and $$v = \frac{\partial \psi}{\partial x} \quad \text{or} \quad \frac{\partial \psi}{\partial x} = c(a^2+x^2-y^2) \longrightarrow (4)$$

Integrating (3) w.r.t. y $\psi = -cxy^2 + \phi(x,t) \longrightarrow (5)$

Diff. (5) partially w.r.t. x

$$\frac{\partial \psi}{\partial x} = -cy^2 + \frac{\partial \phi}{\partial x} \longrightarrow (6)$$

from (4) and (6)

$$-cy^2 + \frac{\partial \phi}{\partial x} = c(a^2+x^2-y^2)$$

$$\Rightarrow \frac{\partial \phi}{\partial x} = c(a^2+x^2)$$

$$\text{or } \phi(x,t) = c\left(a^2x + \frac{x^3}{3}\right) + \psi(y,t).$$

Putting above value in (5), we get

$$\psi = -cxy^2 + c\left(a^2x + \frac{x^3}{3}\right) + \psi(y,t)$$

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